Random band matrix approach to chaotic scattering: the average S-matrix and its pole distribution.

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Random band matrices relevant for open chaotic systems are introduced and studied. The scattering model based on such matrices may serve for the description of preequilibrium chaotic scattering. In the limit of a large number of open channels we calculate the average S-matrix and S-matrix's pole distribution which are found to reduce to those of the full matrix (GOE) case under proper renormalization of the energy scale and strength of coupling to the continuum.

1. It is now generally acknowledged that the Random Matrix Theory (RMT) provides an adequate tool for the description of statistical properties of chaotic quantum systems [1]– [3]. The most of works in the RMT deals with the Hermitian matrices which are, strictly speaking, appropriate for the description of closed systems. Meanwhile, any exited state of a quantum system has a finite lifetime, decaying eventually into open channels. It is well known [4]– [7] that the openness of a system can be incorporated into consideration by means of the effective nonhermitian Hamiltonian

$$\mathcal{H}_{nm} = H_{nm} - i \sum_{c\,(open)} V_n^c V_m^c \ . \qquad (1)$$

Due to coupling to the continuum, the effective Hamiltonian (1) acquires, apart from the intrinsic part H, the antihermitian part which consists of the sum of products of the transitions amplitudes V_n^c between N internal ($|n\rangle$) and M open channel ($|c\rangle$) states. For the T-invariant theory these amplitudes as well as H can be chosen to be real, the matrix $\mathcal{H}$ being symmetric. The eigenvalues $\mathcal{E}_n = E_n - \frac{i}{2}\Gamma_n$ of $\mathcal{H}$ are the complex energy levels of an unstable system, with E_n and Γ_n being, respectively, the energy and width of the n-th level.

Assuming the intrinsic dynamics to be chaotic, the hermitian part of $\mathcal{H}$ is usually supposed to belong to the Gaussian Orthogonal Ensemble (GOE) of random matrices. As to the coupling amplitudes, they are considered to be either fixed [5,6] or random [7]. Resulting generalization of the RMT on the unstable systems proved [5]–[10] to be useful for the description of statistical properties of quantum chaotic scattering. The resonance part of the scattering matrix is represented (see, e.g. [4]) as

$$S_{ab}(E) = \delta_{ab} - 2i \sum_{nm} V_n^a [(E-\mathcal{H})^{-1}]_{nm} V_m^b \ . \qquad (2)$$

in terms of the effective Hamiltonian (1) which describes the evolution of the unstable system formed at the intermediate stage of the collision. Therefore, fluctuations in scattering reflect statistical properties of the complex energy levels of this intermediate system.

Recently, the considerable progress has been achieved in the study of dynamical and statistical properties of S-matrix's poles and their connection to those of scattering [7] - [10]. The limit of very large number of equivalent channels has a special interest since it can be related to the semiclassical approach [11,9]. The noteworthy property of the pole distribution was found [8,9] in this limit: there always exists a finite gap between the upper edge of the distribution of complex energies of resonances and the real energy axis. This gap turns out to be the important characteristic of local fluctuations in chaotic scattering [9,10].

The Gaussian distribution of the matrix elements of the internal Hamiltonian H implies the invariance with respect to the choice of the intrinsic basis. However, the mean-field basis seems to play an exceptional role [12] in many-body chaotic systems. The realistic Hamiltonian matrix has a banded structure in such a representation [13,14]. In fact, Wigner was the first who considered band matrices in connection with properties of (stable) complicated systems [15]. He assumed the nonzero matrix elements to be Gaussian random variables. Now, the theory of random band matrices (RBM) attracts a great interest and is claimed to be relevant for studying quantum chaos (for review see [16]). The most of known analytical results for RBM has been recently obtained by Fyodorov and Mirlin [17,18] who studied RBM in the context of the localization, having reduced the RBM problem to the Efetov's supersymmetric σ-model [19] with the diffusion constant proportional to the square of the bandwidth.

The purpose of this Letter is to extend the RBM approach to the consideration of open chaotic systems described by the effective Hamiltonian (1). The scattering model based on such random matrices can be related to preequilibrium chaotic scattering in a sense intermediate between fully chaotic scattering described by the GOE model and multistep compound reactions [20]. In the "pure" GOE models all internal degrees of freedom being uniformly involved, the intermediate unstable system has already attained its complete thermodynamical equilibrium before a decay takes place. The band structure of H leads to the localization of intrinsic wave functions, an

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intermediate decaying state concluding only a part of the degrees of freedom. Therefore, in addition to the decay time, the new diffusion time scale appears in the problem which characterizes the internal relaxation time. In this Letter we calculate the average S-matrix and S-matrix's pole distribution in the limit of very large number of open equivalent channels.

2. We suppose that the hermitian part of (1) belongs to the Gaussian RBM ensemble defined by

$$\langle H_{nm} H_{nm} \rangle = \lambda^2 J_{nm}(1+\delta_{nm}) \ , \tag{3}$$

where the function $J_{nm} \equiv J(|n-m|)$ decreases sufficiently fast then $|n-m| > b$, with $b \gg 1$ being the effective bandwidth. In the GOE case, when all matrix elements J_{nm} are equal to $\frac{1}{N}$, 2λ determines the radius of Wigner's semicircle low of the eigenvalue distribution [1]. The transition amplitudes V_n^a are considered as fixed quantities subject to the condition [5,6]

$$\sum_n V_n^a V_n^b = \gamma\lambda\delta^{ab} \ , \tag{4}$$

which turns out to be enough for excluding direct reactions. The dimensionless parameter γ characterizes the strength of coupling between the internal motion and channels.

We calculate the average S-matrix and distribution of complex energy levels in the limit $M, N \to \infty$, with $m = M/N < 1$ fixed. It turns out that the band structure of the hermitean part H does not lead to the essential complication when evaluating the one-point average characteristics and all calculations can be done in close analogy with those of the GOE case [8,9]. Below we present rather short description, referring for details to [8,9].

3. The calculation of the average S-matrix is related with that of the average Green function

$$\langle \mathcal{G}(z) \rangle = \left\langle \left(\frac{1}{z-\mathcal{H}} \right) \right\rangle, \quad z = E + i0 \ , \tag{5}$$

which governs the evolution of the intermediate unstable chaotic system. It is convenient to use the following representation for $\langle \mathcal{G}_{nm}(z) \rangle$ [5]

$$\langle \mathcal{G}_{nm}(z) \rangle = 2\frac{\partial}{\partial I_{nm}} \langle \ln Z(z,I) \rangle \Big|_{I=0} \ , \tag{6}$$
$$Z(z,I) = \det(z - H + iVV^T - I)^{-\frac{1}{2}} \ .$$

To carry out averaging in (6) we use the replica method [21,5]. The generating function $Z(z,I)$ can be represented as a multivariable Gaussian integral that makes averaging over the Gaussian RBM ensemble (3) trivial. The further integration can be performed by means of the saddle-point approximation. The saddle-point solution turns out to be stable in the replica space and proportional to $\delta_{nn'}$. Therefore, the replicas decouple and, as was pointed in [8], it is enough to calculate $\ln\langle Z(z,I) \rangle$.

One has

$$\langle Z(z,I) \rangle \cong \int d[\phi] \exp\Big\{ \sum_{n,m=1}^{N} \big[-\frac{\lambda^2}{4} J_{nm} \phi_n^2 \phi_m^2 + \frac{i}{2} \phi_n (z + iVV^T - I)_{nm} \phi_m \big] \Big\} \ , \tag{7}$$

where $d[\phi]$ means the product of differentials and above equality is valid to irrelevant constant. To make the integration over ϕ doable, we introduce, following [8,9], new variables $\sigma_n = \lambda\phi_n^2$ with the help of δ-functions defined by their Fourier representation

$$\delta(\sigma_n - \lambda\phi_n^2) = \frac{1}{\pi} \int d\hat{\sigma}_n \exp\{ -\frac{i}{2} (\sigma_n \hat{\sigma}_n - \lambda\phi_n \hat{\sigma}_n \phi_n) \} \tag{8}$$

After the Gaussian integration over ϕ being done, the subsequent integration can be performed in the saddle-point approximation justified by two large parameters $N, b \gg 1$. We find after some algebra that the average Green function is determined in the following way

$$\langle \mathcal{G}_{nm}(z) \rangle = \big[(z + \lambda\hat{\sigma}_{s.p.} + iVV^T)^{-1} \big]_{nm} \tag{9}$$

by the solution of the saddle-point equations (with respect to σ and $\hat{\sigma}$) for the "Lagrangian" $\mathcal{L}$

$$\mathcal{L} = -\frac{1}{4} \sum_{nm} J_{nm} \sigma_n \sigma_m - \sum_n (\frac{i}{2} \sigma_n \hat{\sigma}_n + \ln(z + \lambda\hat{\sigma}_n)) - \frac{1}{2} \mathrm{tr}_c \ln[1 + iV^T (z + \lambda\hat{\sigma})^{-1} V] \ , \tag{10}$$

where the diagonal matrix $\hat{\sigma} = \mathrm{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_N)$. Note that the trace in the last term in (10) runs over the channel (M-dimensional) space.

Similar to the pure RBM case [17], the saddle point equations possess the translation invariant (independent of n) solution provided that the relation (4) is fulfilled. Going along the same line as in [8,9], we arrive at

$$\langle \mathcal{G}_{nm}(z) \rangle = \frac{\delta_{nm}(z - J_0\lambda^2 g(z) + i\gamma\lambda) - i\sum_c V_n^c V_m^c}{(z - J_0\lambda^2 g(z))(z - J_0\lambda^2 g(z) + i\gamma\lambda)} \ . \tag{11}$$

Here the notation $J_0 \equiv \sum_r J(|r|)$ is introduced. Due to the translation invariance mentioned above the average Green function depends on the only combination of transition amplitudes. This fact makes the average S-matrix be diagonal and equal to

$$\langle S^{ab}(z) \rangle = \delta^{ab} \frac{z - J_0\lambda^2 g(z) - i\gamma\lambda}{z - J_0\lambda^2 g(z) + i\gamma\lambda} \ . \tag{12}$$

The function $g(z)$ denotes the trace of the average Green function which is found to satisfy the cubic equation

$$g(z)\big(J_0\lambda^2 g(z) - z\big) + 1 + \frac{im\gamma\lambda}{J_0\lambda^2 g(z) - z - i\gamma\lambda} = 0 \ , \tag{13}$$

where the (unique) solution with a negative imaginary part has to be chosen [9].

For a closed system ($\gamma = 0$), this cubic equation is reduced to the quadratic one which determines the density of the RBM eigenvalues to be given by Wigner's semicircle law [22] with the half-radius

$$\widetilde{\lambda} = \lambda\sqrt{J_0}\ , \tag{14}$$

renormalized by the factor $\sqrt{J_0}$ (this factor reduces to unity in the case of the GOE). For an open system, additional rescaling of the coupling constant

$$\widetilde{\gamma} = \frac{\gamma}{\sqrt{J_0}}, \tag{15}$$

reduces eqs.(11)–(13) to corresponding ones [23,9] for the full matrix case.

4. The calculation carried out above is valid only for the upper half of the complex plane where the Green function is analytical, all S-matrix singularities being located into the lower part (see eq.(2)). The analyticity is broken where the density of complex energies differs from zero. A special regularization procedure of the pole singularities has been proposed in [8] to calculate the average S-matrix's pole distribution in the lower half-plane of the complex variable $z = x + iy$. Basing on a convenient electrostatic analogy [7,8], the distribution of complex levels is considered to be the source of the two-dimensional electrostatic field with the potential

$$\Phi(x,y) = \frac{1}{N}\langle \ln\det\big\{(z^* - \mathcal{H}^\dagger)(z-\mathcal{H}) + \delta\big\}^{-1}\rangle\ . \tag{16}$$

The limit $\delta \to 0^+$ should be taken at the very end of calculations. Applying the two-dimensional Laplacian to Φ, one gets the average density of complex levels ("charges")

$$4\pi\rho(x,y) = -\Delta\Phi(x,y)\ . \tag{17}$$

The replica trick can be used again for performing the ensemble averaging. The infinitesimal positive δ makes the matrix in the rhs of eq.(16) positive definite for any z. Therefore, the determinant may be represented as the Gaussian integral over a complex N-vector $\psi(1)$. To make the ensemble averaging possible, the product $(z^* - \mathcal{H}^\dagger)(z-\mathcal{H})$ is decoupled [8,9] with the help of the complex Habbard-Stratonovich transformation (N-vector $\psi(2)$). One finally arrives at the integral representation

$$\exp\{N\Phi\} \cong \int d[\psi]\,\big\langle \exp\{i\psi^\dagger\mathcal{M}\psi'\}\big\rangle \tag{18}$$

where $2N\times 2N$ matrix $\mathcal{M}$ is defined as

$$\mathcal{M} = \begin{bmatrix} z^* - \mathcal{H}^\dagger & i\delta \\ i & z-\mathcal{H}\end{bmatrix}\ ,$$

introduced $2N$-vectors being $\psi^T = \big(\psi(1)^T, \psi(2)^T\big)$ and $\psi'^T = \big(\psi(2)^T, \psi(1)^T\big)$.

Performing the same steps as in [8,9], we find the potential $N\Phi$ to be determined by the saddle-point value of the "Lagrangian"

$$\mathcal{L} = -\frac{1}{2}\sum_{nm} J_{nm}\mathrm{tr}_\alpha(\sigma_n\sigma_m) - i\sum_n \mathrm{tr}_\alpha(\sigma_n\hat{\sigma}_n) \tag{19}$$
$$-\sum_n \mathrm{tr}_\alpha \ln(z_\delta + \lambda\hat{\sigma}_n) - \mathrm{tr}_{\alpha c}\ln\big[1 - iV^T(z_\delta + \lambda\widehat{\Sigma})^{-1}Vl\big]\ .$$

Each of N matrices σ_n in (19) has the following structure

$$\sigma = \begin{pmatrix} w & u \\ v & w^*\end{pmatrix}$$

with the real positive u and v whereas $\hat{\sigma}_n$ stands for its Fourier counterpart. We have also introduced $2N\times 2N$ block-diagonal matrices z_δ obtained from $\mathcal{M}$ by setting there $\mathcal{H}$ equal to zero and $\widehat{\Sigma}_{\alpha\beta\ nm} = (\hat{\sigma}_n)_{\alpha\beta}\delta_{nm}$, the $2M\times 2M$ block-diagonal matrix l is unity in the channel subspace and equals to $l = \mathrm{diag}(1,-1)$ in the replica subspace, and V is the unit matrix for replica indices and corresponds to V_n^c for others. In the saddle-point we have the translation invariant saddle-point equations

$$\begin{aligned}&\hat{\sigma} = iJ_0\sigma\\ &\frac{i\sigma}{\lambda}(z_\delta + i\lambda J_0\sigma) + 1 - i\frac{m\gamma\lambda l}{z_\delta + i\lambda J_0\sigma - i\gamma\lambda l} = 0\end{aligned} \tag{20}$$

which differ from the corresponding equations for the full matrix case only in appearing the renormalizing factor J_0. Therefore, the explicit solution can be found in our case in the same way as it has been done in [9]. As a result, one concludes that all poles (charges) lie in the finite domain of the lower part ($y < 0$) of the complex energy plane defined by the condition $x^2 \le \mathsf{x}^2(y)$ with

$$\mathsf{x}^2(y) = -\frac{4m\gamma\lambda^3 J_0}{y} - \Big[\frac{m\lambda^2 J_0}{y} + \frac{1-m}{\gamma\lambda + y}\lambda^2 J_0 - \gamma\lambda\Big]^2\ . \tag{21}$$

Inside this region the density of complex energy levels is equal to

$$4\pi\rho(x,y) = \frac{m}{y^2} + \frac{1-m}{(\gamma\lambda + y)^2} - \frac{1}{J_0\lambda^2}\ . \tag{22}$$

One can easily see that under rescaling (14),(15) the average complex level distribution reduces again to that [9] of the full matrix case.

In conclusion, the band structure of the hermitean part H results only in the renormalization of the energy scale λ (14) and coupling constant γ (15) as compared to the GOE model [9]. In particular, the condition of the "width collectivization" [7,23], $\widetilde{\gamma} \sim 1$, implies again the natural physical condition of the average partial width being comparable with the average level spacing [23]. One should expect nontrivial consequences of the band structure to appear only while considering the higher correlation functions.

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